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ON THE CONVERGENCE OF SEQUENCES OF THE MOREAU-YOSIDA FUNCTION WITH TOW VARIABLES IN THE REFLIXIVE BANACH SPACES

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Abstract: **The aim of this paper is to study and generalize some results and properties with the Moreau-Yosida function of one variable for convex functions to the Moreau-Yosida function of two variables for convex-concave functions. And studing the convergenc of Moreau-Yosida sequence functions according to the concept of Mosco-epi/hypo graph and $\rho-$Housdorff distance in reflixive Banach spaces.**

Keywords: **epi-graph, optimization problems, convex-concave functions, Moreau-yosida distance, parent functions, Mosco-epi/hypo-graph.**

1. INTRODUCTION

Morau-Yosida approximation plays an important role in optimization, because of it's important properties in related of minimization problems.

Many researches contributed to the development of this concept and studying the basic properties of Moreau-Yosida function on lower semi continous convex functions [5,6]. Among many beautiful and important properties of Moreau-Yosida approximation that it is allows us to study nonsmooth functions and turning it into a smooth functions wich have analytical properties to resolve the problem at hand or to identify more than a set of properties.

Studying of Morau-Yosida approximates of colsed convex function falls under the frame work in the study of the stsbility of the solution for optimization problems, wich has implications in model formulation, optimality characterizations, approximation theory. But one of the reasons that there are essentially no global results is that ther did not seem to exist a good metric, i.e. , one with the appropriate theoretical properties and reasonably easy to compute, that could be used to measure the distance between two optimization problems.

The main propose of this article is to study and generalized some properties of Moreau-Yosida function with two variables of convex-concave functions and studying convergence according to concept of Mosco epi/hypo graphical and $\rho-$ Housdoroff distance (epi/hypo distance) for a sequence of convex-concave functions and a sequence of Moreau-Yosida corresponding it wich was studied by many researches in Hilbert spaces, and generalized this results in reflixive Banach spaces. This paper is organized as follows. In section 1, we show notations, definitions and some results in convex analysis. In section 2, we show notations and definitions in convex-concave analysis that we use it in our article. In section 3, we give the premier main results wich include : in theorem 3.1 assume that $K\left(u_0,.\right)\leq\alpha_0\left\|.-v_0\right\|^2+\alpha_1$

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and $K(.,v_0) \geq -\alpha_0 \|u_0 - .\|^2 - \alpha_1$ for some $\alpha_0 \geq 0$ and $\alpha_1 \geq 1$ then $K_{\lambda,\mu}$ is a continuous locally Lipschitz function in $X \times Y$ and is finit valued for all $0 < \lambda < \frac{1}{4\alpha_0}$ and $0 < \mu < \frac{1}{4\alpha_0}$. In theorem 3.2 we show that ther is equivalent between convergence a sequence of convex-concave functions and a sequence of Moreau-Yosida functions corresponding it according to the concept of Mosco in reflixive Banach spaces. In theorems 3.4 and 3.5 we studying the relationship between the epi/hypo distance and Moreau-Yosida distane wich was introduced by Attouh and Wets [4] but for convex-concave functions. Finally we proved that $\lim_{n\to\infty} H_\rho\left(L^n_{\lambda,\mu}, L_{\lambda,\mu}\right) = 0$ iff $L^n \xrightarrow{M-e/h} L$ by using support functions for nonempty convex sets $epiF_\lambda$ (epigraphical parent function for Moreau-Yosida function $L_{\lambda,\mu}$) in reflixive Banach spaces.

2. NOTATIONS AND DEFINITIONS IN CONVEX ANALYSIS

Let $(X, \|.\|)$ be a normed linear space and $(X^*, \|.\|_*)$ its dual, the duality pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x, x^* \rangle$, and let $f : X \to \overline{R}$ of the real valued extension function defined on X, we well denote the set of the real valued extended functions defined on X by $\overline{R}^X$. For a function $f \in \overline{R}^X$ the set :

$$epi\, f = \{ (x, \lambda) \in X \times R \;/\; f(x) \leq \lambda \}$$

is called the epigraph of f, and f is called convex lower semicontinuous if its epigraph is a convex (closed) subset of $X \times R$. Furthermore, f is called proper if its epigraph nonempty, or if its domain is nonempty

$$dom\, f = \{ x \in X \;/\; f(x) < +\infty \} \neq \phi$$

$\Gamma(X)$ well denote the proper, lower semi continuous convex functions defined on X

- For $f \in \Gamma(X)$, its *conjugate* $f^* \in \Gamma^*(X^*)$ is defined by the familiar formula :

$$f^*(x^*) = \sup_{x\in X} \{ <x, x^*> - f(x) \}, \forall x^* \in X^* \quad (1)$$

- Moreau-Yosida approximation of f of parameter λ where $\lambda > 0$ is defined by :

$$f_\lambda(x) = \inf_{u\in X} \left\{ f(u) + \frac{1}{2\lambda}\|x-u\|^2 \right\}$$

These approximate play an important role in the analysis of variational limit problem, basically bacause a sequence of functions $\{f^n : X \to]-\infty, +\infty], n = 1, ...\}$ epi-convergence(with respect to the strong topology of X) to the lower semicontinuous function f if anf only if

$$f = \sup_{\lambda>0} \limsup_{n\to\infty} f^n_\lambda = \sup_{\lambda>0} \liminf_{n\to\infty} f^n_\lambda$$

- Let $\{f^n : X \to \overline{R}, n \in N\}$ be a sequence of functions. We say that f is the *Mosco-epi-limit* of this sequence, if for all x in X :

$$i)\ \forall (x_n)_{n\in N}\ ; x_n \underset{n}{\overset{w}{\to}} x : f(x) \leq \liminf_n f_n(x_n)$$

$$ii)\ \exists (\zeta_n)_{n\in N}\ ; \zeta_n \underset{n}{\overset{s}{\to}} \zeta : f(\zeta) \geq \limsup_n f_n(\zeta_n)$$

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Then we can write $f = Mosco - epi \lim_{n} f_n$ or $f_n \xrightarrow{M} f$ where $s\,(w)$ strong(resp.weak) tobology on X

- Let X be normed linear space and $\{ f, g : X \to \overline{R} \}$ two functions in X then for all $\rho \geq 0$, $\lambda > 0$ we can define Moreau –Yosida distance between two proper functions f, g :

$$d_{\lambda,\rho}(f,g) = \sup_{\|x\| \leq \rho} |f_\lambda(x) - g_\lambda(x)|$$

- $\rho - Housdoroff$ *distances* on X [3,4,5]

For each $\rho \geq 0$, ρB denotes the closed ball of radius ρ ; and for any subset C of X , we define C_ρ by :

$$C_\rho : = C \cap \rho B$$

For any pair C and D of subsets of X , the *Housdoroff excess* of C over D is defined by :

$$haus_\rho(D, C) = Sup\{ e(C_\rho, D) ; e(D_\rho, C) \}$$

Where $e(C, D) = \sup_{x \in C} d(x, D)$; $(e(C, D) = 0 \ if \ C = \phi)$

and for all $\rho \geq 0$, the $\rho - Housdoroff$ *distances* between C and D is defined by :

$$haus_\rho(D, C) = Sup\{ e(C_\rho, D) ; e(D_\rho, C) \}$$

A sequence of sub sets $(D_n)_{n \in N}$ of X , is said to converge with respect to the $\rho - Housdoroff$ *distances* to some D iff for all $\rho \geq 0$,

$$\lim_{n \to \infty} haus_\rho(D_n, D) = 0$$

- $\rho - Housdoroff$ *distances* on $\overline{R}^X$ **[5]**

For all $\rho \geq 0$, the $\rho - Housdoroff$ *distances* between two functions $f, g \in \overline{R}^X$ is defined by :

$$H_\rho(f, g) = haus_\rho(epi\ f, epi\ g)$$

where $epi\ f\ and\ epi\ g$ are two subsets of $X \times R$, and the ball of $X \times R$ is the set:

$$\rho B_{X \times R} = \{ (x, \alpha) \in X \times R \ / \ \|x\| \leq \rho \ , \ |\alpha| \leq \rho \}$$

A sequence of functions $(f_n)_{n \in N}$ of $\overline{R}^X$, is said to converge with respect to the $\rho - Housdoroff$ *distances* to some f iff for all $\rho \geq 0$,

$$\lim_{n \to \infty} H_\rho(f_n, f) = 0$$

The concept $\rho - Housdoroff$ *distances* is also called the ρ *- epigraphical distance*

Proposition 1.1 [8]

Suppose X be a reflixive Banach space and $\{ f_n, f ; n \in N \}$ be a sequence of functions in $\Gamma(X)$ .Then for all

$$\rho \geq 0 \text{ if } \lim_{n \longrightarrow \infty} h_\rho(f_n, f) = 0 \text{ then } f_n \xrightarrow{M} f$$

Proposition 1.2 [5]

Suppose C and D are nonempty subsets of Banach space X with norm $\|\,.\,\|$ such that $haus_{\rho}(D,C)$ is finite. Then

$$haus_{\rho}(C,D)=\sup_{x\in X}\left|dist(x,C)-dist(x,D)\right|$$

Proposition 1.3 [5]

Suppose C and D are nonempty closed convex subsets of reflixive Banach space X with norm $\|\,.\,\|$ such that $haus_{\rho}(D,C)$ is finite. Then

$$haus_{\rho}(C,D)=\sup_{\|v\|\le 1}\left|\Psi_C^*(v)-\Psi_D^*(v)\right|$$

3. NOTATIONS AND DEFINITIONS IN CONVEX-CONCAVE ANALYSIS

Let X, Y, X^*, Y^* be linear spaces such that X (resp. Y) is in separate

duality with X^* (resp. Y^*) via pairings denoted by $\langle .,. \rangle$.

Let us consider $L : X \times Y^* \to \overline{R}$, wich is convex in the x variable and concave in the y variable and we define :

$$F : X \times Y^* \to \overline{R}$$
$$G : X^* \times Y \to \overline{R}$$

By

$$F(x,y^*)=\sup_{y\in Y}\left\{L(x,y)+\langle y,y^*\rangle\right\}$$
$$G(x^*,y)=\inf_{x\in X}\left\{L(x,y)-\langle x,x^*\rangle\right\}$$

F (resp. G) is the convex(resp.concave) parent of the convex-concave function L . A function L is said to be closed if it's parents are conjugate each other i.e. $F^*=-G$ and $-G^*=F$

For closed convex-concave function L the associated equivalence class is an interval denoted by $\left[\underline{L},\overline{L}\right]$ with

$$\underline{L}(x,y)=\sup_{x^*\in X^*}\left\{G(x^*,y)+<y,y^*>\right\}$$

$$\overline{L}(x,y)=\inf_{y^*\in Y^*}\left\{F(x,y^*)-<y,y^*>\right\}$$

If we denote by $\Gamma(X \times Y)$ the class of all convex l.s.c functions defined on $X \times Y$ with values in $\overline{R}$

Mosco-epi/hypo-convergenc [3] :

Let X and Y be reflixive Banach spaces and $\left\{L^n, L : X \times Y^* \to \overline{R}\right\}$ a collection of convex-concave bivariate functions, we say that L^n Mosco-epi/hypo-convergence to L if :

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$$\text{i)}\ \ \forall (x,y)\in X\times Y,\forall y_n \xrightarrow[n]{w} y,\exists x_n \xrightarrow[n]{s} x:\ \limsup_n L_n(x_n,y_n)\le L(x,y)$$

$$\text{ii)}\ \ \forall (x,y)\in X\times Y,\forall x_n \xrightarrow[n]{w} x,\exists y_n \xrightarrow[n]{s} y:\ \liminf_n L_n(x_n,y_n)\ge L(x,y)$$

Moreau-Yosida approximate of closed concex-concave function [3] :

Let X and Y be reflixive Banach spaces and $L: X\times Y^{*}\to \overline{R}$ closed convex-concave function, for all $\lambda>0$, $\mu>0$

$$L_{\lambda,\mu}(x,y):=\inf_u \sup_v \left\{ L(u,v)+\frac{1}{2\lambda}\|x-u\|^2-\frac{1}{2\mu}\|y-v\|^2, u\in X, v\in Y \right\} \quad (2)$$

$L_{\lambda,\mu}$ is called Moreau-Yosida approximation of idex λ,μ of L , this function has a unique saddle point $(x_{\lambda,\mu},y_{\lambda,\mu})$ characterized by

$$\left(\frac{x-x_{\lambda,\mu}}{\lambda},-\frac{y-y_{\lambda,\mu}}{\mu}\right)\in \partial L(x_{\lambda,\mu},y_{\lambda,\mu})$$

And the first drivative is defined by :

$$\nabla L_{\lambda,\mu}(x,y)=\left(\frac{x-x_{\lambda,\mu}}{\lambda},-\frac{y-y_{\lambda,\mu}}{\mu}\right)$$

Moreau-yosida monotone is represented by :

$$A^{k}_{\lambda,\mu}(x,y)=\left(\frac{x-x_{\lambda}}{\lambda},\frac{y-y_{\mu}}{\mu}\right)$$

Moreau-Yosida distance function [3] :

Let us consider $\{L,K: X\times Y\to \overline{R}\}$ two convex-concave cloced functions for all $\lambda>0$, $\mu>0$ and $\rho\ge 0$, then Moreau-Yosida distance between L and K is defuned by :

$$d_{\lambda,\mu,\rho}(K,L)=\sup_{\substack{\|x\|\le\rho \\ \|y\|\le\rho}}\left|K_{\lambda,\mu}(x,y)-L_{\lambda,\mu}(x,y)\right|$$

A sequence of convex-concave functions $(K_n)_{n\in N}$ of $X\times Y$, is said to converge with respect to the Moreau-Ysida distance to K iff for all $\rho\ge 0$, and $\lambda>0,\mu>0$

$$\lim_{n\to\infty} d_{\lambda,\mu,\rho}(K_n,K)=0$$

$\rho-$Housdorff distances on $\overline{R}^{X\times Y}$ [3]:

For all $\rho\ge 0$, the $\rho-$Housdorff distances between two functions $L,K\in \overline{R}^{X\times Y}$ is defined by :

$$H_{\rho}(L,K)=h_{\rho}(F,\Phi) \quad (3)$$

Where F and Φ are parents convex functions for convex-concave functions L and K .

A sequence of functions $(L_n)_{n\in N}$ of $\overline{R}^{X\times Y}$, is said to converge with respect to the $\rho - Housdoroff$ *distances* to some f iff for all $\rho \geq 0$,

$$\lim_{n\to\infty} H_\rho(K_n, K) = 0$$

Proposition 2.1 [12] :

Let X and Y be reflixive Banach spaces and $\{L^n, L : X\times Y^* \to \overline{R}\}$ a collection of closed convex-concave bivariate functions, F (resp. F^n) is the convex parents functions of the convex-concave functions L (resp. L^n), then the folloeing statements are equivalent:

$$i)\ F^n \xrightarrow{M} F$$

$$ii)\ L^n \xrightarrow{M-e/h} L$$

$$iii)\ L^n_\mu(., y^*) \xrightarrow{M} L_\mu(., y^*)$$

4. THE MAIN RESULTS

Theorem 3.1 :

Suppose X and Y are two normed linear space and $K : X\times Y \to \overline{R}$ is a proper function with two variables, suppose there is $(u_0, v_0) \in X\times Y$, $\alpha_0 \geq 0$ and $\alpha_1 \geq 1$ such that

$$K(u_0,.) \leq \alpha_0 \|.-v_0\|^2 + \alpha_1 \quad (4)$$

$$K(.,v_0) \geq -\alpha_0 \|u_0 - .\|^2 - \alpha_1 \quad (5)$$

For all $0 < \lambda < \frac{1}{4\alpha_0}$ and $0 < \mu < \frac{1}{4\alpha_0}$ then :

1- $K_{\lambda,\mu}$ is a finit valued

2- $K_{\lambda,\mu}$ is locally Lipschitz, i.e.

$$\|K_{\lambda,\mu}(x_1, y_1) - K_{\lambda,\mu}(x_2, y_2)\| \leq C(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

Where C the constant Lipschitz.

Proof :

1- From definition of $K_{\lambda,\mu}$ in (2) we can write

$$K_{\lambda,\mu}(x,y) = \inf_{u\in X} \sup_{v\in y} \left\{ K(u,v) + \frac{1}{2\lambda}\|x-u\|^2 - \frac{1}{2\mu}\|y-v\|^2 \right\}$$
$$\leq \sup_{v\in Y} \left\{ \alpha_0 \|v - v_0\|^2 + \frac{1}{2\lambda}\|x - u_0\|^2 - \frac{1}{2\mu}\|y - v\|^2 \right\} + \alpha_1$$

By using $\|v - v_0\|^2 \leq \frac{1}{2}\left(\|v - y\|^2 + \|y - v_0\|^2\right)$ then we find

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$$K_{\lambda,\mu}(x,y)\le \sup_{v\in y}\left\{\frac{\alpha_0}{2}\|v-y\|^2+\frac{\alpha_0}{2}\|y-v_0\|^2+\frac{1}{2\lambda}\|x-u_0\|^2-\frac{1}{2\mu}\|y-v\|^2\right\}+\alpha_1$$

Hence for all $0<\mu<\dfrac{1}{4\alpha_0}$ then we can write

$$K_{\lambda,\mu}(x,y)\le \sup_{v}\left\{\frac{\alpha_0}{2}\|v-y\|^2+\frac{\alpha_0}{2}\|y-v_0\|^2+\frac{1}{2\lambda}\|x-u_0\|^2-\frac{\alpha_0}{2}\|y-v\|^2\right\}+\alpha_1$$

$$K_{\lambda,\mu}(x,y)\le \frac{1}{2\lambda}\|x-u_0\|^2+\frac{\alpha_0}{2}\|y-v_0\|^2+\alpha_1 \quad (6)$$

On the other hand, for all $0<\lambda<\dfrac{1}{4\alpha_0}$ we can prove in the same way that

$$K_{\lambda,\mu}(x,y)\ge -\frac{1}{2\mu}\|y-v_0\|^2-\frac{\alpha_0}{2}\|x-u_0\|^2-\alpha_1 \quad (7)$$

From (6) and (7) we can find that $K_{\lambda,\mu}$ is a finit valued

2- In the begening we will prove that the application $y\to K_{\lambda,\mu}(x,y)$ is locally Lipschitz, for this reason we will choosing λ and μ in the too little values such

that $K_{\lambda,\mu}(.,.)$ is a finit valued.

Let $y\in Y$ and $\varepsilon>0$ there is $v_\varepsilon=v_\varepsilon(\lambda,\mu,x)$ such that

$$K_{\lambda,\mu}(x,y)\ge \inf_{u\in X}\left\{K(u,v_\varepsilon)+\frac{1}{2\lambda}\|x-u\|^2-\frac{1}{2\mu}\|y-v_\varepsilon\|^2\right\}\ge K_{\lambda,\mu}(x,y)-\varepsilon \ (8)$$

From (7) , (8) and for $u=u_0$ this yields

$$K(u_0,v_\varepsilon)+\frac{1}{2\lambda}\|x-u_0\|^2-\frac{1}{2\mu}\|y-v_\varepsilon\|^2+\varepsilon\ge -\frac{1}{2\mu}\|y-v_0\|^2-\frac{\alpha_0}{2}\|x-u_0\|^2-\alpha_1 \quad (9)$$ since

$K(u,.)\le \alpha_0\|.-v_0\|^2+\alpha_1$ we have

$$\alpha_0\|v_\varepsilon-v_0\|^2+\alpha_1+\frac{1}{2\lambda}\|x-u_0\|^2-\frac{1}{2\mu}\|y-v_\varepsilon\|^2+\varepsilon\ge -\frac{1}{2\mu}\|y-v_0\|^2-\frac{\alpha_0}{2}\|x-u_0\|^2-\alpha_1 \quad (10)$$ by using

$\|v_\varepsilon-v_0\|^2\le \frac{1}{2}\left(\|v_\varepsilon-y\|^2+\|y-v_0\|^2\right)$ then we find

$$\alpha_0\|v_\varepsilon-v_0\|^2\le \frac{\alpha_0}{2}\left(\|v_\varepsilon-y\|^2+\|y-v_0\|^2\right)\le \frac{\alpha_0}{2}\|v_\varepsilon--y\|^2+\frac{\alpha_0}{2}\|y-v_0\|^2$$

This yields

$$2\alpha_1+\left(\frac{1}{2\lambda}+\frac{\alpha_0}{2}\right)\|x-u_0\|^2+\varepsilon+\frac{\alpha_0}{2}\|v_\varepsilon-y\|^2+\frac{\alpha_0}{2}\|y-v_0\|^2-\frac{1}{2\mu}\|y-v_\varepsilon\|^2\ge -\frac{1}{2\mu}\|y-v_0\|^2$$

$$\left(\frac{1}{2\lambda}+\frac{\alpha_0}{2}\right)\|x-u_0\|^2+\left(\frac{1}{2\mu}+\frac{\alpha_0}{2}\right)\|y-v_0\|^2+2\alpha_1+\varepsilon\ge\left(\frac{1}{2\mu}-\frac{\alpha_0}{2}\right)\|y-v_\varepsilon\|^2$$

$$\|y-v_\varepsilon\|^2 \le \left(\frac{2\mu}{1-\alpha_0\mu}\right)(2\alpha_1+\varepsilon)+\frac{\mu}{\lambda}\left(\frac{\alpha_0+\lambda}{1-\mu\alpha_0}\right)\|x-u_0\|^2+\left(\frac{1+\mu\alpha_0}{1-\mu\alpha_0}\right)\|y-v_0\|^2 \quad (11)$$

since (8) is true for all y then it's true for all $y=y_1$ and $y=y_2$ then we can write

$$K_{\lambda,\mu}(x,y_1)\ge \inf_{u\in X}\left\{K(u,v_\varepsilon)+\frac{1}{2\lambda}\|x-u\|^2\right\}-\frac{1}{2\mu}\|y_1-v_\varepsilon\|^2+\frac{1}{2\mu}\|y_2-v_\varepsilon\|^2-\frac{1}{2\mu}\|y_2-v_\varepsilon\|^2$$

$$\ge K_{\lambda,\mu}(x,y_2)-\varepsilon+\frac{1}{\mu}\left(\frac{1}{2}\|y_2-v_\varepsilon\|^2-\frac{1}{2}\|y_1-v_\varepsilon\|^2\right)$$

We use the convexity of $t\to\frac{1}{2}t^2$ on R^+, and the subgradiant to obtain

$$\frac{1}{2}\left(\|y_1-y_2\|+\|y_2-v_\varepsilon\|\right)^2-\frac{1}{2}\|y_2-v_\varepsilon\|^2\le\left(\|y_1-y_2\|+\|y_2-v_\varepsilon\|\right)\left(\|y_1-y_2\|\right)$$

$$K_{\lambda,\mu}(x,y_2)-K_{\lambda,\mu}(x,y_1)\le\varepsilon+\frac{1}{\mu}\left(\|y_1-y_2\|+\|y_2-v_\varepsilon\|\right)\left(\|y_1-y_2\|\right) \quad (12)$$

From (11) , (12) and let $\varepsilon\to 0$. This yields

$$K_{\lambda,\mu}(x,y_2)-K_{\lambda,\mu}(x,y_1)\le C_{y_2}.\mu^{-1}\|y_2-y_1\| \quad (13)$$

Where

$$C_{y_2}=\left\{\|y_1-y_2\|+\left[\frac{2\mu}{1-\alpha_0}(2\alpha_1+\varepsilon)+\frac{\mu}{\lambda}\left(\frac{\alpha_0+\lambda}{1-\mu\alpha_0}\right)\|x-u_0\|^2+\left(\frac{1+\mu\alpha_0}{1-\mu\alpha_0}\right)\|y-v_0\|^2\right]^{\frac{1}{2}}\right\}$$

Is a constant that depends only on $\|y_2-y_1\|$, $\|x-u_0\|$, λ , μ , α_1 , α_0 , $\|y_2-v_0\|$

Interchanging the role of y_1 and y_2, we obtain a similar inequality with a constant C_{y_1}. Setting $C_1=\max(C_{y_1},C_{y_2})$ yields the inequality

$$\left|K_{\lambda,\mu}(x,y_2)-K_{\lambda,\mu}(x,y_1)\right|\le\frac{1}{\mu}C_1\|y_2-y_1\| \quad (14)$$

By using (15),(16) and in the similar way we can prove that $x\to K_{\lambda,\mu}(x,y)$ is locally Lipschitz and it follows

$$\left|K_{\lambda,\mu}(x_2,y)-K_{\lambda,\mu}(x_1,y)\right|\le\frac{1}{\lambda}C_2\|x_2-x_1\| \quad (15)$$

Where $C_2=\max\{C_{x_1},C_{x_2}\}$ and

$$C_{x_2}=\left\{\|x_1-x_2\|+\left[\frac{2\lambda}{1-\alpha_0\lambda}(2\alpha_1+\varepsilon)+\frac{\lambda}{\mu}\left(\frac{\alpha_0+\mu}{1-\lambda\alpha_0}\right)\|y-v_0\|^2+\left(\frac{1+\lambda\alpha_0}{1-\lambda\alpha_0}\right)\|x-u_0\|^2\right]^{\frac{1}{2}}\right\}$$

from (13) and (14) we find

$$\left|K_{\lambda,\mu}(x_1,y_1)-K_{\lambda,\mu}(x_2,y_2)\right|\le\left|K_{\lambda,\mu}(x_1,y_1)-K_{\lambda,\mu}(x_1,y_2)\right|+\left|K_{\lambda,\mu}(x_1,y_2)-K_{\lambda,\mu}(x_2,y_2)\right|$$

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$$\leq C\left(\|y_1 - y_2\| + \|x_1 - x_x\|\right)$$

Where $C = \max\left\{\frac{C_1}{2\mu}, \frac{C_2}{2\lambda}\right\}$ ∎

Proposition 3.2 :

Let X and Y be reflixive Banach spaces and $\left\{K^n, K : X \times Y^* \to \overline{R}\right\}$ a collection of closed convex-concave bivariate functions, then the following statements are equivalent

$$i)\ K^n \xrightarrow{M-e/h} K$$

$$ii)\ \lim_{n\to\infty} K^n_{\lambda,\mu}(x,y) = K_{\lambda,\mu}(x,y) \quad ; \forall\ \lambda,\mu > 0\ ; \forall (x,y) \in X \times Y$$

Proof :

Suppose $L(x,y) = K(u,v) + \frac{1}{2\lambda}\|u-x\|^2 - \frac{1}{2\mu}\|v-y\|^2$ $i) \Rightarrow ii)$

$$L_n(x,y) = K^n(u,v) + \frac{1}{2\lambda}\|u-x\|^2 - \frac{1}{2\mu}\|v-y\|^2$$

The sequence $\left\{L_n, L : X \times Y \to \overline{R}\right\}$ is closed convex-concave functions

We will prove that $L_n \xrightarrow{M-e/h} L$, and for this reason we will prove Mosco-epi/hypo-conditions :

1)- for $(x,y) \in X \times Y$ and for any $y_{\lambda,\mu} \xrightarrow{w} y$ there exist $x_{\lambda,\mu} \xrightarrow{s} x$ such that

$$\limsup_{n\to\infty} L_n(x_{\lambda,\mu}, y_{\lambda,\mu}) \leq L(x,y)$$

2)- for $(x,y) \in X \times Y$ and for all $x_{\lambda,\mu} \xrightarrow{s} x$ there exist $y_{\lambda,\mu} \xrightarrow{w} y$ such that

$$\liminf_{n\to\infty} L_n(x_{\lambda,\mu}, y_{\lambda,\mu}) \geq L(x,y)$$

Lets prove the first condition for that we will take $\sup_v$ for $L_n(x, y_{\lambda,\mu})$, then we have

$$\sup_v L_n(x, y_{\lambda,\mu}) = \sup_v\left\{K^n(u,v) + \frac{1}{2\lambda}\|x-u\|^2 - \frac{1}{2\mu}\|y_{\lambda,\mu} - v\|^2\right\}$$

$$\leq \sup_v\left\{\overline{K}^n(u,v) - \frac{1}{2\mu}\|y_{\lambda,\mu} - v\|^2\right\} + \frac{1}{2\lambda}\|x-u\|^2$$

$$\leq \sup_v\left\{\inf_{y^*}\left\{F^n(u,y^*) - \langle y^*, v\rangle\right\} - \frac{1}{2\mu}\|y_{\lambda,\mu} - v\|^2\right\} + \frac{1}{2\lambda}\|x-u\|^2$$

$$\leq \sup_v \inf_{y^*}\left\{F^n(u,y^*) - \langle y^*, v\rangle - \frac{1}{2\mu}\|y_{\lambda,\mu} - v\|^2\right\} + \frac{1}{2\lambda}\|x-u\|^2$$

$$\leq \inf_{y^*}\left\{F^n(u,y^*) + \sup_v\left\{\langle y^*, v\rangle - \frac{1}{2\mu}\|y_{\lambda,\mu} - v\|^2\right\}\right\} + \frac{1}{2\lambda}\|x-u\|^2$$

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Suppose $z = y_{\lambda,\mu} - v$ and $x = u$, and by using (1)

$$\sup_{v} L_n\left(x, y_{\lambda,\mu}\right) \le \inf_{y^*}\left\{F^n\left(x, y^*\right) - < y^*, y_{\lambda,\mu} > + \frac{\mu}{2}\left\|y^*\right\|^2\right\}$$

let $\mu \to 0$ and $n \to \infty$ This yields :

$$\lim_{n\to\infty}\sup L_n\left(x, y_{\lambda,\mu}\right) \le \lim_{n\to\infty}\inf\left\{F^n\left(x, y^*\right) - < y, y^* >\right\}$$

$$\le \inf\left\{\lim_{n\to\infty} F^n\left(x, y^*\right) - < y, y^* >\right\} = \overline{L}\left(x, y\right)$$

and this true for all $y_{\lambda,\mu} \xrightarrow{w} y$ and for $x : x_{\lambda,\mu} \xrightarrow{s} x$ this yields

$$\lim_{n\to\infty}\sup L_n\left(x_{\lambda,\mu}, y_{\lambda,\mu}\right) \le \overline{L}\left(x, y\right)$$

Now Lets prove the second condition for that we will take $\inf_{u}$ for $L_n\left(x_{\lambda,\mu}, y\right)$, then we have

$$\inf_{u} L_n\left(x_{\lambda,\mu}, y\right) = \inf_{u}\left\{K^n(u,v) + \frac{1}{2\lambda}\left\|x_{\lambda,\mu} - u\right\|^2 - \frac{1}{2\mu}\left\|y - v\right\|^2\right\}$$

$$\ge \inf_{u}\left\{\underline{K}^n(u,v) + \frac{1}{2\lambda}\left\|x_{\lambda,\mu} - u\right\|^2\right\} - \frac{1}{2\mu}\left\|y - v\right\|^2$$

$$\ge \inf_{u}\left\{\sup_{x^*}\left\{G^n\left(x^*, v\right) + < x^*, x >\right\} + \frac{1}{2\lambda}\left\|x_{\lambda,\mu} - u\right\|^2\right\} - \frac{1}{2\mu}\left\|y - v\right\|^2$$

$$\ge \sup_{x^*}\left\{G^n\left(x^*, v\right) + \inf_{u}\left\{< x^*, x > + \frac{1}{2\lambda}\left\|x_{\lambda,\mu} - u\right\|^2\right\}\right\} - \frac{1}{2\mu}\left\|y - v\right\|^2$$

Suppose $z = x_{\lambda,\mu} - u$ and $y = v$, and by using (1)

$$\inf_{u} L_n\left(x_{\lambda,\mu}, y\right) \ge \sup_{x^*}\left\{G^n\left(x^*, v\right) + < x^*, x_{\lambda,\mu} > - \frac{\lambda}{2}\left\|x^*\right\|^2\right\}$$

let $\lambda \to 0$ and $n \to \infty$ This yields :

$$\lim_{n\to\infty}\inf L_n\left(x_{\lambda,\mu}, y\right) \ge \lim_{n\to\infty}\sup\left\{G^n\left(x^*, y\right) + < x^*, x_{\lambda,\mu} >\right\}$$

$$\ge \sup\left\{G\left(x^*, y\right) + < x^*, x_{\lambda,\mu} >\right\} = \underline{L}\left(x_{\lambda,\mu}, y\right)$$

and this true for all $x : x_{\lambda,\mu} \xrightarrow{s} x$ and for $y : y_{\lambda,\mu} \xrightarrow{w} y$ this yields

$$\lim_{n\to\infty}\sup L_n\left(x_{\lambda,\mu}, y_{\lambda,\mu}\right) \le \overline{L}\left(x, y\right)$$

Conditions of convergence achieved,(see Aze [2, Theorem 2.5]) we find that the point $\left(x_{\lambda,\mu}, y_{\lambda,\mu}\right)$ is convergence to the point (x, y) wich form a single saddle point for the function L, for any $L_n \in \left[\underline{L}_n, \overline{L}_n\right]$ and $L \in \left[\underline{L}, \underline{L}\right]$ this yields

$$L_n\left(x_{\lambda,\mu}, y_{\lambda,\mu}\right) \xrightarrow[n\to\infty]{} L\left(x, y\right) \text{ for all } x_{\lambda,\mu} \xrightarrow{s} x \text{ and } y : y_{\lambda,\mu} \xrightarrow{w} y \text{ wich mean}$$

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$$\left\{K^n(u,v)+\frac{1}{2\lambda}\|x-u\|^2-\frac{1}{2\mu}\|y-v\|^2\right\}\xrightarrow[n\to\infty]{}\left\{K(u,v)+\frac{1}{2\lambda}\|x-u\|^2-\frac{1}{2\mu}\|y-v\|^2\right\}$$

By taking $\inf_u \sup_v$ we find the convergence is true $K_{\lambda,\mu}(x,y)=\lim_{n\to\infty}K^n(x_{\lambda,\mu},y_{\lambda,\mu})$

$$ii\,)\Rightarrow i\,)$$

let's consider the augmented Lagrangians $\psi^n(u)=K_\mu^n(u,y^*)$ and $\psi(u)=K(u,y^*)$, from proposition 2.1 we know that

$$\psi^n\xrightarrow{M}\psi\Leftrightarrow K_\mu^n(.,y^*)\xrightarrow{M}K_\mu(.,y^*)$$

The Moreau-yosida approximation $\psi_\lambda^n(x)$ for $\lambda>0$ is defined by

$$\psi_\lambda^n(x)=\inf_{u\in X}\left\{\psi^n(u)+\frac{1}{2\lambda}\|x-u\|^2\right\}$$

The augmented Lagrangians for function K^n is defined by

$$K_\mu^n(u,y^*)=\sup_{\eta^*\in Y^*}\left\{K(u,\eta^*)-\frac{1}{2\mu}\|y^*-\eta^*\|^2\right\}$$

Wich mean that

$$\psi_\lambda^n(x)=\inf_{u\in X}\left\{\sup_{\eta^*\in Y^*}\left\{K(u,\eta^*)-\frac{1}{2\mu}\|y^*-\eta^*\|^2\right\}+\frac{1}{2\lambda}\|x-u\|^2\right\}$$

$$=\inf_{u\in X}\sup_{\eta^*\in Y^*}\left\{K(u,\eta^*)-\frac{1}{2\mu}\|y^*-\eta^*\|^2+\frac{1}{2\lambda}\|x-u\|^2\right\}=K_{\lambda,\mu}^n(x,y^*)$$

Let $n\to\infty$, we derive that $ii\,)\Rightarrow i\,)$ ■

Proposition 3.3 :

Let X and Y are two normed linear space and $L,K:X\times Y\to\overline{R}$ are convex-concave proper functions, then for $(u_0,v_0)\in X\times Y$, $\alpha_0\geq 0$ and $\alpha_1\in R$ such that

$$L(u_0,.)\leq\alpha_0\|.-v_0\|^2+\alpha_1$$

$$L(.,v_0)\geq-\alpha_0\|u_0-.\|^2-\alpha_1$$

For all $0<\lambda<\frac{1}{4\alpha_0}$ and $0<\mu<\frac{1}{4\alpha_0}$, $\lambda>0$, $\mu>0$

Then $$H_\rho(L_{\lambda,\mu},K_{\lambda,\mu})\leq H_\gamma(L,K)$$

Where the constant γ , that depends on ρ

Proof:

$L_{\lambda,\mu}$ and $K_{\lambda,\mu}$ are Moreau-Yosida approximates of index λ,μ of L and K respectively, F_λ and Φ_λ are convex

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parent functions of $L_{\lambda,\mu}$ and $K_{\lambda,\mu}$ resp.

F and Φ are convex and proper functions, implies that $epiF \neq \phi$ and $epi\,\Phi \neq \phi$

To have $h_\beta(F,\Phi) \leq \eta$ means that $(epiF)_\beta \subset \eta(epi\,\Phi)$ where

$\eta D := \{x \mid d(x,D) \leq \eta\}$ is the $\eta-$fattening of D. By adding $epi\,\frac{1}{2\lambda}\|.\|^2$ it follows

$(epiF)_\beta + epi\,\frac{1}{2\lambda}\|.\|^2 \subset \eta(epi\,\Phi) + epi\,\frac{1}{2\lambda}\|.\|^2$, and this inclusion with

$epi\,\Phi + epi\,\frac{1}{2\lambda}\|.\|^2 \subset epi\,\Phi_\lambda$ yields $(epiF)_\beta + epi\,\frac{1}{2\lambda}\|.\|^2 \subset \eta(epi\,\Phi_\lambda)$

Since $epi_s F_\lambda = epi_s F + epi_s\left(\frac{1}{2\lambda}\right)\|.\|^2$ where $epi_s F$ is the strict epigraph of F, is defined by

$epi_s F := \{(x,\alpha) \mid F(x) < \alpha\}$ .wich mean that

$$\left(epi_s F_\lambda + epi_s\,\frac{1}{2\lambda}\|.\|^2\right)_\rho \subset (epiF)_\beta + epi\,\frac{1}{2\lambda}\|.\|^2 \Rightarrow \left(epi_s F_\lambda + epi_s\,\frac{1}{2\lambda}\|.\|^2\right)_\rho \subset \eta(epi\,\Phi_\lambda)$$

$$\Rightarrow \mathrm{e}\left((epi_s F_\lambda)_\rho + epi\,\Phi_\lambda\right)_\rho \leq \eta$$

For all $\varepsilon > 0$ and for all $h_\beta(F,\Phi) < \eta$ this yields $\mathrm{e}\left((epi_s F_\lambda)_{\rho-\varepsilon} + epi\,\Phi_\lambda\right)_\rho \leq \eta + \varepsilon$

The asserted inequality $H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu}) \leq H_\gamma(L,K)$ now follows from the fact that $L_{\lambda,\mu}$ is locally Lipschitz and L,K are summetric roles This mean $h_\rho(F_\lambda,\Phi_\lambda) \leq \eta + \varepsilon$. Let $\varepsilon \to \infty$ we drive that $H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu}) \leq \eta$ ■

Modified Kenmochi conditions :

The Kenmochi conditions provid a partical criterion for computing, or at least estimating, the epi-distance between two functions

We have modified the conditions to fit our search for estimating the epi/hypo-distance on convex-concave functions

Suppose K, L are proper closed convex-concave functions defined on $X \times Y$, for some $\alpha_0 \geq 0$ and $\alpha_1 \in R$, and for all $(u_0, v_0) \in X \times Y$ such that

$$L(u_0,.) \leq \alpha_0 \|.-v_0\|^2 + \alpha_1$$

$$L(.,v_0) \geq -\alpha_0 \|u_0 - .\|^2 - \alpha_1$$

$L_{\lambda,\mu}$ and $K_{\lambda,\mu}$ are Moreau-Yosida approximates of index λ,μ of L and K

respectively, F_λ and Φ_λ are convex parent functions of $L_{\lambda,\mu}$ and $K_{\lambda,\mu}$ resp.

F and Φ are convex and proper functions, implies that $epiF \neq \phi$ and $epi\,\Phi \neq \phi$

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Then the folloewing conditions to be called the modified Kenmochi conditions:

(a) For all $\rho > \rho_0$ and $(x_1, y_1) \in X \times Y$ such that $|K(x_1, y_1)| \le \rho$ and $L(x_1, y_1) \le \rho$

For every $\varepsilon > 0$ there is exists $(x_2, y_2) \in X \times Y$ that satisfies :

$$\|x_1 - x_2\| + \|y_1 - y_2\| \le H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu}) + \varepsilon$$
$$K_{\lambda,\mu}(x_2, y_2) \le L_{\lambda,\mu}(x_1, y_1) + H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu}) + \varepsilon$$

(b) For all $\rho > \rho_0 > 0$ there exists a constant $\eta(\rho) \in R_+$ depending on ρ, such that for all $|L(x_1, y_1)| \le \rho$ and $\|x_1\| \le \rho$, $\|y_1\| \le \rho$ there is exists $(x_2, y_2) \in X \times Y$ that satisfies:

$$\|x_1 - x_2\| + \|y_1 - y_2\| \le \eta(\rho)$$
$$K_{\lambda,\mu}(x_2, y_2) \le L_{\lambda,\mu}(x_1, y_1) + \eta(\rho)$$

This mean that to compute $H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu})$ we have to find the best constant $\eta(\rho)$ and in the symmetric cindition , then with $\rho_1 := \rho + \alpha_0 \rho^2 + |\alpha_1|$ this satisfies

$$H_\rho(L_{\lambda,\mu}, K_{\lambda,\mu}) \le \eta(\rho_1)$$

Proposition 3.4 :

Suppose X and Y are two normed linear space and $L, K : X \times Y \to \overline{R}$ is a proper function with two variables, suppose there is $(u_0, v_0) \in X \times Y$, $\alpha_0 \ge 0$ and $\alpha_1 \ge 1$ such that

$$K(u_0, .) \le \alpha_0 \|. - v_0\|^2 + \alpha_1$$

$$K(., v_0) \ge -\alpha_0 \|u_0 - .\|^2 - \alpha_1$$

For all $0 < \lambda < \dfrac{1}{4\alpha_0}$, $0 < \mu < \dfrac{1}{4\alpha_0}$ and $\rho \ge 0$ then :

$$d_{\lambda,\mu,\rho}(K, L) \le \beta(\lambda, \mu, \rho).H_{\gamma(\lambda,\mu,\rho)}(L_{\lambda,\mu}, K_{\lambda,\mu})$$

Proof:

The functions $L_{\lambda,\mu}$ and $K_{\lambda,\mu}$ are finit values, and locally Lipschtiz. This can be to conclude that whenever $\|(x_1, y_1)\| \le \rho$ and $\|(x_2, y_2)\| \le \rho$, both $K_{\lambda,\mu}(x, y)$ and $L_{\lambda,\mu}(x, y)$ are bounded in absolute value by ρ'

$$\rho' \ge \max\left\{\left|K_{\lambda,\mu}(u_0, v_0) + \frac{1}{2\lambda}C_\rho.\rho - \frac{1}{2\mu}C_\rho.\rho\right|, \left|L_{\lambda,\mu}(u_0, v_0) + \frac{1}{2\lambda}C_\rho.\rho - \frac{1}{2\mu}C_\rho.\rho\right|\right\}$$

Where C_ρ Lipschtiz constant. Setting $\rho_1 = \max[\rho, \rho']$, and let us estimate $K_{\lambda,\mu}(x_1, y_1) - L_{\lambda,\mu}(x_1, y_1)$ by using the modified Kenmochi conditions, for all $\varepsilon > 0$ there exist (x_2, y_2) such that

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$$\|x_1 - x_2\| + \|y_1 - y_2\| \le H_{\rho_1}\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) + \varepsilon$$

$$K_{\lambda,\mu}(x_2, y_2) \le L_{\lambda,\mu}(x_1, y_1) + H_{\rho_1}\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) + \varepsilon$$

$$K_{\lambda,\mu}(x_1, y_1) - L_{\lambda,\mu}(x_1, y_1) = K_{\lambda,\mu}(x_1, y_1) - K_{\lambda,\mu}(x_2, y_2) + K_{\lambda,\mu}(x_2, y_2) - L_{\lambda,\mu}(x_1, y_1)$$
$$\le C\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right) + H_{\rho_1}\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) + \varepsilon$$
$$\le (C+1) H_{\rho_1}\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) + (C+1).\varepsilon$$

Let $\varepsilon \to 0$ and $\beta := C + 1$ where C is constant depending only on λ, μ is defined by $C = \frac{1}{2\lambda} C_{\rho_2} - \frac{1}{2\mu} C_{\rho_2} + 1$.

where C_{ρ_2} Lipschtiz constant and $\rho_2 := \rho + H_{\rho_1}\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) + 1$

$$d_{\lambda,\mu,\rho}(L, K) = \sup\left|L_{\lambda,\mu}(x, y) - K_{\lambda,\mu}(x, y)\right| \le \beta . H_\gamma(L, K)$$

$$\Rightarrow d_{\lambda,\mu,\rho}(L, K) \le \beta . H_\gamma\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) \blacksquare$$

Proposition 3.5 :

Suppose X and Y are two normed linear space and $L, K : X \times Y \to \overline{R}$ is a proper function with two variables, F (resp. Φ) is the convex parent of the convex-concave function L (resp. K) such that $epiF \ne \phi$ and $epi\,\Phi \ne \phi$ for $\lambda > 0$, $\rho > 0$, $\mu = \frac{1}{\lambda}$ then

$$H_\rho\left(L_{\lambda,\mu}, K_{\lambda,\mu}\right) \le d_{\lambda,\mu,\rho}(L, k)$$

Proof:

From definition of Moreau-Yosida distance for convex-concave functions and their parent convex functions we can write

$$d_{\lambda,\rho}(F, \Phi) = d_{\lambda,\mu,\rho}(K, L) \text{ and } d_{\lambda,\rho}(F, \Phi) = \sup_{\substack{\|x\| \le \rho \\ \|y\| \le \rho}} \left|F_\lambda(x, y') - \Phi_\lambda(x, y')\right|$$

Foe all $\lambda > 0$, $\rho > 0$, $\mu = \frac{1}{\lambda}$ then $K_{\lambda,\frac{1}{\lambda}}(x, y)$ is convex-concave function and $F_\lambda(x, \lambda y)$ is parent convex function such that

$$K_{\lambda,\frac{1}{\lambda}}(x, y) = F_\lambda(x, \lambda y) - \frac{\lambda}{2}\|y\|^2 \ , \ y = \lambda y'$$

if $\Phi_\lambda(x, y') \le \eta$, $\|x\| \le \rho$ then $F_\lambda(x, y') - d_{\lambda,\rho}(F, \Phi) \le \eta$, this implies that

$$\left(\eta + d_{\lambda,\rho}(F, \Phi), (x, y')\right) \in epiF_\lambda$$

$$\Rightarrow e\left(\left(epi\,\Phi_\lambda\right)_\rho, \left(epiF_\lambda\right)\right) \le d_{\lambda,\rho}(F, \Phi)$$

$$\Rightarrow h_\rho(F_\lambda, \Phi_\lambda) = H_\rho\left(K_{\lambda,\mu}, L_{\lambda,\mu}\right) \le d_{\lambda,\rho}(F, \Phi) = d_{\lambda,\mu,\rho}(K, L) \blacksquare$$

$\rho -$ Housdorff distances and support functions:

Many reaserchs have been studing the relationship between indicator and support functions and $\rho -$ Housdorff distance for covex functions.

The conjugate of the indicator function Ψ_S of a set S is the support function of S denoted by $\Psi_S^*(v)$, i.e $\Psi_S^*(v) = \sup\left\{\langle v, x\rangle \mid x \in S\right\}$

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$epiF$ and $epi\,\Phi$ are nonempty convex subsets of reflixive Banach spaces X and Y , the Hausdorff distance between $epiF$ and $epi\,\Phi$ is given by

$$haus_{\rho}\left(epiF,epi\,\Phi\right):=\sup\left[\sup_{x\in epi\,\Phi} dist\left(x,epiF\right), \sup_{x\in epiF} dist\left(x,epi\,\Phi\right)\right] \quad (16)$$

Where $dist\left(x,epiF\right)=\inf_{y\in epiF}\left\|x-y\right\|$

F_{λ} is a finit valued $F_{\lambda}\left(x,.\right)\le 2\alpha_0\left\|x-.\right\|^2+\alpha_1$ and is locally Lipschitz, $epiF_{\lambda}$ is epigraph of F_{λ} and it's convex closed set. As with the distance function $d_{\lambda,\mu,\rho}$ generated by the Moreau-Yosida approximates we can define a Hausdorff metric on the spaces of closed convex sets $epiF_{\lambda}$. It's mean that we can build a distance functon on the space of proper lower semicontinuous convex function F_{λ} . We can write support function for $epiF_{\lambda}$ in the form

$$\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)=\sup\left[\left\langle v,x\right\rangle+\alpha.\beta \mid F_{\lambda}\left(x,.\right)<\alpha\right] \quad ; \quad \alpha=2\alpha_0\left\|x-.\right\|^2+\alpha_1$$

$$=\begin{cases}\infty & ;\beta>0\\ \sup\left\langle v,x\right\rangle & ;\beta=0\\ \sup\left[\left\langle v,x\right\rangle-\left(-\beta\right)F\left(x,.\right)\right] & ;\beta<0\end{cases}$$

In other formulation we can write (16) according to convex parent functions :

$$haus_{\rho}\left(epiF_{\lambda},epi\,\Phi_{\lambda}\right):=\inf\left[\theta \mid \sup_{x\in epi\,\Phi_{\lambda}} dist\left(x,epiF_{\lambda}\right)\le\theta, \sup_{x\in epiF_{\lambda}} dist\left(x,epi\,\Phi_{\lambda}\right)\le\theta\right]$$

Where

$$\sup_{x\in epi\,\Phi_{\lambda}} dist\left(x,epiF_{\lambda}\right)=\sup_{x\in epi\,\Phi_{\lambda}}\sup_{\|v\|_{*}\le 1}\left|<x,v>-\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)\right|$$
$$=\sup_{\|v\|_{*}\le 1}\left|\sup_{x\in epi\,\Phi_{\lambda}}<v,x>-\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)\right|$$
$$=\sup_{\|v\|_{*}\le 1}\left|\Psi^{*}_{epi\,\Phi_{\lambda}}\left(v,\beta\right)-\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)\right|$$

Wich mean that $haus_{\rho}\left(epiF_{\lambda},epi\,\Phi_{\lambda}\right)=\sup_{\|v\|\le\rho}\left|\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)-\Psi^{*}_{epi\,\Phi_{\lambda}}\left(v,\beta\right)\right|$

This yields

$$\begin{aligned}H_{\rho}\left(L_{\lambda,\mu},K_{\lambda,\mu}\right)=h_{\rho}\left(F_{\lambda},\Phi_{\lambda}\right)=haus_{\rho}\left(epiF_{\lambda},epi\,\Phi_{\lambda}\right)\\ =\sup_{\|v\|\le\rho}\left|\Psi^{*}_{epiF_{\lambda}}\left(v,\beta\right)-\Psi^{*}_{epi\,\Phi_{\lambda}}\left(v,\beta\right)\right|\end{aligned} \quad (17)$$

Proposition 3.6 :

Let X and Y be reflixive Banach spaces and $\left\{L_n,L:X\times Y\to\overline{R}\right\}$ a collection of closed convex-concave bivariate functions.

For all $\lambda>0$, $\mu>0$ and $\rho\ge 0$, the following statements are equivalent :

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$i)\ H_\rho\left(L^n_{\lambda,\mu}, L_{\lambda,\mu}\right) \xrightarrow[n\to\infty]{} 0$

$ii)\ L^n_{\lambda,\mu} \xrightarrow[n\to\infty]{} L_{\lambda,\mu}$

Where $L_{\lambda,\mu}$, $L^n_{\lambda.\mu}$ are Moreau-Yosida approximates of index λ, μ of L , L^n

respectively, F^n_λ and F_λ are convex parent functions of $L^n_{\lambda,\mu}$ and $L_{\lambda,\mu}$ resp.

Proof:

$i) \Rightarrow ii)$ Simply use Propositions 1.1 and 2.1 and relation between $H_\rho\left(L^n_{\lambda,\mu}, L_{\lambda,\mu}\right)$ and $h_\rho\left(F^n_\lambda, F_\lambda\right)$ we derive that $\lim_{n\to\infty} L^n_{\lambda,\mu}(x,y) = L_{\lambda,\mu}(x,y)$, because of $L_{\lambda,\mu}$, $L^n_{\lambda,\mu}$ are locally continues Lipschitz [18] on bouded sets

$ii) \Rightarrow i)$ since $L^n_{\lambda,\mu} \xrightarrow[n\to\infty]{} L_{\lambda,\mu}$ then $L^n \xrightarrow{M-e/h} L$ (proposition 3.2) this mean that $F^n \xrightarrow{M} F$ (proposition 3.2) and $F^n_\lambda \xrightarrow{M} F_\lambda$ where :

$F_\lambda\left(x, y^*\right) = \sup_{y\in Y}\left\{L_{\lambda,\mu}(x,y) + \left\langle y, y^*\right\rangle\right\}$ and $F^n_\lambda\left(x_n, y^*\right) = \sup_{y_n\in Y}\left\{L_{\lambda,\mu}(x_n, y_n) + \left\langle y_n, y^*\right\rangle\right\}$

By taking limit to $\Psi^*_{epiF^n_\lambda}(v,\beta) = \sup\left[\left\langle v, x_n\right\rangle + \alpha.\beta \mid F^n_\lambda(x,.) < \alpha\right]$ when $n \to \infty$, and since the convergence according to the Mosco concept is true, this mean that there is a sequence x_n strongly converget to x i.e $\left\langle v, x_n\right\rangle \xrightarrow[n\to\infty]{} \left\langle v, x\right\rangle$, this yields

$\Psi^*_{epiF^n_\lambda} \xrightarrow[n\to\infty]{} \Psi^*_{epiF_\lambda}$ and from (17) we derive that $H_\rho\left(L^n_{\lambda,\mu}, L_{\lambda,\mu}\right) \xrightarrow[n\to\infty]{} 0$ ■

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